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Automata Theory Based on Quantum Logic. (I)

Mingsheng Ying[1]

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We present a basic framework of automata theory based on quantum logic. In particular, we introduce the orthomodular lattice-valued (quantum) predicate of recognizability and establish some of its fundamental properties.

1. INTRODUCTION

It is well known that almost all mathematical theories such as group theory and topology are based on classical (Boolean) logic and intuitionistic mathematics is built on intuitionistic logic. One may naturally conceive of the problem whether we are able to establish some mathematical theories based on other nonclassical logics besides intuitionistic logic. Indeed, as early as 1952, Rosser and Turquette [RT52] proposed the following problem: if there are many-valued theories beyond the level of predicate calculus, then what are the details of such theories? This problem was thought of by them as one of the major unsolved problems in many-valued logics. Recently, an attempt has been made by the author in [Y91–93, Y93] to give a partial and elementary answer in the case of point-set topology to the question raised above. I used a semantical analysis method to develop topology based on residuated lattice-valued logic, especially continuous-valued logic, and initiated a new approach to topology in fuzzy set theory.

Similarly, various models of computations are investigated in the framework of classical logic; more explicitly, all properties of these models of computations are deduced by classical logic as their (meta)logical tool. Then we may also ask what are the similarities and differences between the properties of the models of computations in classical logic and the corresponding

[1] State Key Laboratory of Intelligent Technology and Systems, Department of Computer Science and Technology, Tsinghua University, Beijing 100084, China; e-mail: yingmsh@mail.tsinghua.edu.cn.

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ones in nonclassical logics. There has been a very big population of nonclassical logics; of course it is unnecessary to construct models of computations in each nonclassical logic and compare them with the ones in classical logic because some nonclassical logics are completely irrelevant to the behavior of computations. Nevertheless, as will be explained shortly, it is worth studying deeply and systematically models of computations based on quantum logic.

Quantum logic was introduced by Birkhoff and von Neumann [BN36] in the thirties as the logic of quantum mechanics. The starting point of quantum logic is von Neumann's Hilbert space formalism of quantum mechanics; closed subspaces of a Hilbert space are identified with propositions concerning a quantum mechanical system and their suitable lattice operations are treated as connectives, and this leads directly to an orthomodular lattice. Nowadays, what is usually called quantum logic in the mathematical physics literature is not truly logic, but quite often refers to the theory of orthomodular lattices. There is also another, much more 'logical' point of view on quantum logic in which quantum logic is seen as a logic whose truth values range over an orthomodular lattice; for an excellent exposition of the latter treatment of quantum logic, see Dalla Chiara [DC86]. It might have seemed that both points of view on quantum logic have no obvious links to computations, but the appearance of the idea of quantum computers changed this situation dramatically.

Quantum computers were first envisaged by Feynman [F82, F86] and elaborated and formalized by Deutsch in [D85]. In particular, Deutsch [D85] proposed that quantum computers might be able to perform certain types of computations that classical computers can only perform very inefficiently. One of the most striking advances was made by Shor [S94], who discovered a polynomial-time algorithm on quantum computers for prime factorization which is a central problem in computer science and of which the best known algorithm on classical computers is exponential. Since then quantum computation has been an extremely exciting and rapidly growing field of research. While models of classical computers were developed based on Boolean logic, Vedral and Plenio [VP98] advocated that quantum computers require quantum logic, something fundamentally different from classical Boolean logic. As mentioned above, quantum logic has existed for a long time, and the issue is how to apply quantum logic in the analysis and design of quantum computers.

Automata are simple theoretical models of computers. The existing theory of automata is built upon Boolean logic, and so it might not be suitable for quantum computers that obey logical laws different from that in Boolean logic. The purpose of this paper and its continuations is to establish a theory of automata based on quantum logic, with the hope that the results gained in our approach may offer new insights into quantum computation. Our main technique is the so-called semantical analysis method, which was first adopted in the author's previous works [Y91, Y92a,b, Y91–93, Y93, Y94] and which, roughly speaking, transforms the intended conclusion (usually represented as an implication formula) into an inequality in the lattice of truth values and then proves this inequality in an algebraic way.

2. SEMANTICS OF QUANTUM LOGIC

This section is a preliminary one in which we recall some basic notions and results and to fix notations on the semantic aspect of quantum logic; for more details, we refer to [DC86]. In this paper, quantum logic is understood as a complete orthomodular lattice-valued logic. A complete orthomodular is a 7-tuple $l = \langle L, \leq, \wedge, \vee, \perp, 0, 1\rangle$, where:

(1) $\langle L, \leq, \wedge, \vee, 0, 1\rangle$ is a complete lattice, 0 and 1 are the least and greatest elements of L, respectively, $\leq$ is the partial ordering in L, and for any $M \subseteq L$, $\wedge M$ and $\vee M$ stand for the greatest lower bound and the least upper bound of M, respectively.

(2) $\perp$ is a unary operation on L, called orthocomplement, and required to satisfy the following conditions:

(2.1) $a \wedge a^{\perp} = 0$, $a \vee a^{\perp} = 1$.

(2.2) $a^{\perp\perp} = a$.

(2.3) $a \leq b$ implies $b^{\perp} \leq a^{\perp}$.

(2.4) $a \wedge (a^{\perp} \vee (a \wedge b)) \leq b$.

To make a complete orthomodular lattice available as the set of truth values of a logic, we need to define a binary operation, called an implication operator, on it such that this operation may serve as the interpretation of implication in this logic. Unfortunately, all implication operators that one can reasonably introduce in an orthomodular lattice are more or less anomalous in the sense that they do not share most of the fundamental properties of the implication in classical logic. One relatively reasonable implication operator among them is the Sasaki arrow:

$$a \to b \overset{\text{def}}{=} a^{\perp} \vee (a \wedge b) \qquad \text{for any } a, b \in L$$

which enjoys, among others, the following useful property:

(3) $a \leq b$ iff $a \to b = 1$.

For a detailed discussion of the Sasaki arrow, see Román and Rumbos [RR91] and Román and Zuazua [RZ99]. We finally define the bi-implication operator on l as follows:

$$a \leftrightarrow b \stackrel{\text{def}}{=} (a \to b) \wedge (b \to a) \qquad \text{for any} \quad a, b \in L$$

Given a complete orthomodular lattice $l = \langle L, \leq, \wedge, \vee, \perp, 0, 1\rangle$. An l-valued (quantum) logic possesses a nullary connective **a** for each $a \in L$ as well as two primitive connectives, one unary connective $\neg$ (negation) and one binary connective $\wedge$ (conjunction), and a primitive quantifier $\forall$ (universal quantifier). Beside logical language, in the sequel we will also need some notations such as $\in$ (membership) from set-theoretic language. An l-valued interpretation is an interpretation in which every predicate symbol is associated with a mapping from the universe of discourse into L, i.e., an l-valued relation, and the others are interpreted as usual; for every (well-formed) formula φ, its truth value $\lceil \varphi \rceil \in L$, and the truth valuation rules for logical and set-theoretic formulas are given as follows:

(i) $\lceil \mathbf{a} \rceil = a$.
(ii) $\lceil \neg\varphi \rceil = \lceil \varphi \rceil^{\perp}$.
(iii) $\lceil \varphi \wedge \psi \rceil = \lceil \varphi \rceil \wedge \lceil \psi \rceil$.
(iv) if U is the universe of discourse, then $\lceil (\forall x)\varphi(x) \rceil = \wedge_{u \in U} \lceil \varphi(u) \rceil$.
(v) $\lceil x \in A \rceil = A(x)$.

Here A is a set constant (unary predicate symbol) and it is interpreted as a mapping, also denoted as A, from the universe into L, i.e., an l-valued set (more exactly, an l-valued subset of the universe). Note that in the above truth valuation rules $\wedge$ and $\vee$ on the left-hand side are connectives in quantum logic, whereas $\wedge$ and $\vee$ on the right-hand side stand for operations in the orthomodular lattice l of truth values.

To simplify the notations in what follows, it is necessary to introduce several derived formulas:

(v) $\varphi \vee \psi \stackrel{\text{def}}{=} \neg(\neg\, \varphi \wedge \neg\, \psi)$.
(vi) $\varphi \to \psi \stackrel{\text{def}}{=} \neg\varphi \vee (\varphi \wedge \psi)$.
(vii) $\varphi \leftrightarrow \psi \stackrel{\text{def}}{=} (\varphi \to \psi) \wedge (\psi \to \varphi)$.
(viii) $(\exists x)\varphi \stackrel{\text{def}}{=} \neg(\forall x)\neg\varphi$.
(ix) $A \subseteq B \stackrel{\text{def}}{=} (\forall x)(x \in A \to x \in B)$.
(x) $A \equiv B \stackrel{\text{def}}{=} (A \subseteq B) \wedge (B \subseteq A)$.

As we claimed in the introduction, quantum logic will act as our metalogic in the theory of automata developed in this paper. Then we still have to introduce several metalogical notions for quantum logic. For every orthomodular lattice $l = \langle L, \leq, \wedge, \vee, \perp, 0, 1\rangle$, if Γ is a set of formulas and φ a formula, then φ is a semantic consequence of Γ in l-valued logic, written $\Gamma \stackrel{l}{\vDash} \varphi$, whenever $\wedge_{\psi \in \Gamma} \lceil \psi \rceil \leq \lceil \varphi \rceil$ for all l-valued interpretations. In particular, $\underset{l}{\vDash} \varphi$ means that $\phi \stackrel{l}{\vDash} \varphi$, i.e., $\lceil \varphi \rceil = 1$ always holds for every l-valued interpretation; in other words, 1 is the unique designated truth value in l. Furthermore, if $\Gamma \overset{l}{\models} \varphi$ (resp. $\overset{l}{\models}\varphi$) for all orthomodular lattice l, then we say that φ is a semantic consequence of Γ (resp. φ is valid) in quantum logic and write $\Gamma \models \varphi$ (resp. $\models\varphi$).

3. RECOGNIZABILITY

We first recall some basic notions in classical automata theory. Let Σ be a finite alphabet whose elements are called labels. Then an automaton over Σ is a quadruple $\mathfrak{R} = \langle Q, I, T, E\rangle$ in which:

(i) Q is a finite set whose elements are called states.
(ii) $I \subseteq Q$ and states in I are said to be initial.
(iii) $T \subseteq Q$ and states in T are said to be terminal.
(iv) $E \subseteq Q \times \Sigma \times E$, and each $(p, \sigma, q) \in E$ is called a transition in (or an edge of) $\mathfrak{R}$ and it means that input σ makes state p becomes q.

A path in $\mathfrak{R}$ is a finite sequence of the form $c = q_0\sigma_1 q_1 \ldots q_{k-1}\sigma_k q_k$ such that $(q_i, \sigma_{i+1}, q_{i+1}) \in E$ for each $i < k$. In this case, the sequence $\sigma_1 \ldots \sigma_k$ is called the label of c. A path $c = q_0\sigma_1 q_1 \ldots q_{k-1}\sigma_k q_k$ is said to be successful if $q_0 \in I$ and $q_k \in T$. The behavior of an automaton $\mathfrak{R}$ is the set of labels of all successful paths in $\mathfrak{R}$. Let $A \subseteq \Sigma^* = \cup_{n=0}^{\infty} \Sigma^n$. Then A is recognizable if there is an automaton $\mathfrak{R}$ over Σ such that A is the behavior of $\mathfrak{R}$.

Let $l = \langle L, \leq, \wedge, \vee, \perp, 0, 1\rangle$ be an orthomodular lattice, and let Σ be a finite alphabet. Then an l-valued (quantum) automaton over Σ is a quadruple $\mathfrak{R} = \langle Q, I, T, \delta\rangle$ where Q, I, and T are as in a (classical) automaton and δ is an l-valued subset of $Q \times \Sigma \times Q$ [l-valued (ternary) predicate on Q, Σ and Q], i.e., a mapping from $Q \times \Sigma \times Q$ into L, and called the l-valued (quantum) transition relation of $\mathfrak{R}$; and intuitively, $\delta(p, \sigma, q)$ stands for the truth value (in quantum logic) of the proposition that input σ causes state p to become q.

We write $\mathbf{A}(\Sigma, l)$ for the (proper) class of all l-valued automata over Σ.

Before defining the concept of recognizability for l-valued automata, we need to introduce some auxiliary notions and notations. We set

$$T(Q, \Sigma) = (Q \times \Sigma)^* \times Q = \bigcup_{n=0}^{\infty} [(Q \times \Sigma)^n \times Q]$$

i.e., the set of all alternative sequences of states and labels beginning at a state and also ending at a state. For any $c = q_0\sigma_1 q_1 \ldots q_{k-1}\sigma_k q_k \in T(Q, \Sigma)$, k is the length of c and denoted by $|c|$, q_0 is the beginning of c and denoted by $b(c)$, q_k is the end of c and denoted by $e(c)$, and sequence $s = \sigma_1 \ldots \sigma_k$ is called the label of c and denoted by $lb(c)$.

Now, we are ready to define the key notion—recognizability for l-valued automata—in this paper.

Definition 3.1. Let $\Re \in \mathbf{A}(\Sigma, l)$. Then:

(1) The l-valued (unary) predicate $path_\Re$ on $T(Q, \Sigma)$ is defined as $path_\Re \in L^{T(Q,\Sigma)}$ [the set of all mappings from $T(Q, \Sigma)$ into L]: for every $q_0\sigma_1 q_1 \ldots q_{k-1}\sigma_k q_k \in T(Q, \Sigma)$,

$$path_\Re(q_0\sigma_1 q_1 \ldots q_{k-1}\sigma_k q_k) \overset{\text{def}}{=} \bigwedge_{i=0}^{k-1} [(q_i, \sigma_{i+1}, q_{i+1}) \in \delta]$$

Intuitively, the truth value of the proposition that $q_0\sigma_1 q_1 \ldots q_{k-1}\sigma_k q_k$ is a path in $\Re$ is

$$\lceil path_\Re(q_0\sigma_1 q_1 \ldots q_{k-1}\sigma_k q_k)\rceil = \bigwedge_{i=0}^{k-1} \delta(q_i, \sigma_{i+1}, q_{i+1})$$

(2) The l-valued (unary) predicate $rec_\Re$ on Σ^* is defined as $rec_\Re \in L^{\Sigma^*}$: for every $s \in \Sigma^*$,

$$rec_\Re(s) \overset{\text{def}}{=} (\exists c \in T(Q, \Sigma))(b(c) \in I \wedge e(c) \in T \wedge lb(c) = s \wedge path_\Re(c))$$

Intuitively, the truth value of the proposition that s is recognizable by $\Re$ is

$$\lceil rec_\Re(s)\rceil = \vee\{\lceil path_\Re(c)\rceil: c \in T(Q, \Sigma), b(c) \in I, e(c) \in T \text{ and } lb(c) = s\}$$

$rec_\Re$ is defined above as an l-valued unary predicate on Σ^*, so it may also be seen as an l-valued subset of Σ^*, i.e., $rec_\Re: \Sigma^* \to L$ and $rec_\Re(s) = \lceil rec_\Re(s)\rceil$ for all $s \in \Sigma^*$.

Definition 3.2. The l-valued (unary) predicate Rec_Σ on L^{Σ^*} (the set of all l-valued subsets of Σ^*) is defined as $Rec_\Sigma \in L^{(L^{\Sigma^*})}$: for each $A \in L^{\Sigma^*}$,

$$Rec_\Sigma(A) \overset{\text{def}}{=} (\exists \Re \in \mathbf{A}(\Sigma, l))(A \equiv rec_\Re)$$

In other words, the truth value of the proposition that A is recognizable is

$$\lceil Rec_\Sigma(A)\rceil = \vee\{\lceil A \equiv rec_\Re\rceil: \Re \in \mathbf{A}(\Sigma, l)\}$$

It should be noted that the (automaton) variable $\Re$ bounded by the existential quantifier on the right-hand side of the defining formula of Rec_Σ ranges over the proper class $\mathbf{A}(\Sigma, l)$. The reader familiar with axiomatic set theory may worry that this definition will cause a certain set-theoretic difficulty, but we stay well away from anything genuinely problematic. Indeed, for any l-valued automaton $\Re = \langle Q, I, T, \delta\rangle$, there is a bijection $\varsigma: Q \to |Q|$

(the cardinality of Q) $= \{0, 1, \ldots, |Q| - 1\}$ and we can construct a new l-valued automaton $\varsigma(\mathfrak{R}) = \langle |Q|, \varsigma(I), \varsigma(T), \varsigma(\delta)\rangle$, where $\varsigma(\delta)(m, \sigma, n) = \delta(\varsigma^{-1}(m), \sigma, \varsigma^{-1}(n))$ for any $m, n \in |Q|$ and $\sigma \in \Sigma$. It is easy to see that $rec_{\mathrm{P}} = rec_{\varsigma(\mathfrak{R})}$. Then in Definition 3.2 we may only require that the variable $\mathfrak{R}$ bounded by the existential quantifier ranges over all l-valued automata whose state sets are subsets of ω (the set of all nonnegative integers) and the class of all l-valued automata with subsets of ω as state sets is really a set (and in fact it is a subset of $(2\omega)^3 \times \cup_{Q\subseteq\omega} L^{Q\times\Sigma\times Q}$). In most situations, however, the original version of Definition 3.2 is much more convenient and compatible with the corresponding definition in classical automata theory.

We first give a simple connection between recognizability in classical automata theory and the l-valued predicate Rec_Σ introduced above.

Proposition 3.3. Let $A \subseteq \Sigma^*$ be recognizable (in classical automata theory), $B \in L^{\Sigma^*}$, and $suppB = \{s \in \Sigma^*: B(s) > 0\} \subseteq A$, and let

$$\lambda = \vee\{\wedge_{s\in A}[a \leftrightarrow B(s)]: a \in L\}$$

Then $\overset{l}{\models} \boldsymbol{\lambda} \to Rec_\Sigma(B)$. In particular, if $A \subseteq \Sigma^*$ is recognizable, then for every $\lambda \in L$, $\overset{l}{\models} Rec_\Sigma(A[\lambda])$, where $A[\lambda] \in L^{\Sigma^*}$ is given as

$$A[\lambda](s) = \begin{cases} \lambda & \text{if} \quad s \in A, \\ 0 & \text{otherwise} \end{cases}$$

Proof. Since A is recognizable, there must be an automaton $\mathfrak{R} = \langle Q, I, T, E\rangle$ whose behavior is A. Now, for each $a \in L$, we construct an l-valued automaton $\wp_a = \langle Q, I, T, \delta_a\rangle$ such that

$$\delta_a(p, \sigma, q) = \begin{cases} a & \text{if} \quad (p, \sigma, q) \in E, \\ 0 & \text{otherwise} \end{cases}$$

Then it is easy to know that for all $s \in \Sigma^*$,

$$\lceil rec_{\wp_a}(s)\rceil = \begin{cases} a & \text{if} \quad s \in A, \\ 0 & \text{otherwise,} \end{cases}$$

and $\lceil B \equiv rec_{\wp_a}\rceil = \wedge_{s\in A}[a \leftrightarrow B(s)]$. Therefore, we have $\lceil Rec_\Sigma(B)\rceil \geq \vee\{\lceil B \equiv rec_{\wp_a}\rceil: a \in L\} = \lambda$. ■

We now present an example to demonstrate that the l-valued predicate Rec_Σ defined above is not trivial, that is, in general it does not degenerate into a two-valued (Boolean) predicate.

Example 3.4. We paste together observables of the spin-one-half system and obtain an orthomodular lattice $L(x) \oplus L(\bar{x})$, where $L(x) = \{0, p_-, p_+, 1\}$ corresponds to the outcomes of a measurement of the spin states along the

x axis and $L(\bar{x}) = \{\bar{0} = 1, \overline{p_-}, \overline{p_+}, \bar{1} = 0\}$ is obtained by measuring the spin states along a different spatial direction; $L(x) \oplus L(\bar{x})$ may be visualized as the following "Chinese lantern" (see [Sv98] for a more detailed description of $L(x) \oplus L(\bar{x})$):

$$\begin{array}{ccccc} & & 1 & & \\ p_- & p_+ & & \overline{p_-} & \overline{p_+} \\ & & 0 & & \end{array}$$

By a routine calculation we have $p_- \leftrightarrow p_+ = p_- \leftrightarrow \overline{p_-} = p_- \leftrightarrow \overline{p_+} = 0$ and $p_- \leftrightarrow 1 = p_-$. Thus, for each $\lambda \in L(x) \oplus L(\bar{x})$, $\lambda \not\leq p_-$ implies $p\pm \leftrightarrow \lambda \leq p_-$.

Furthermore, let $\Sigma = \{\sigma, \tau\}$ and $A = \{\sigma^n\tau^n\colon n \in \omega\}$, and for any $t \in L(x) \oplus L(\bar{x})$, let $A_t \in L^{\Sigma^*}$ be given as follows:

$$A_t(s) = \begin{cases} 1 & \text{if} \quad s \in A, \\ t & \text{otherwise} \end{cases}$$

Then it holds that $\overset{l}{\models} \mathbf{p}_- \leftrightarrow Rec_\Sigma(A_{p_-})$, i.e., $\lceil Rec_\Sigma(A_{p_-})\rceil = p_-$. In fact, we know that Σ^* is recognizable ([E74], Example II.2.3), and with Proposition 3.3 it is easy to see that $\lceil Rec_\Sigma(A_{p_-})\rceil \geq p_-$. Conversely, for any l-valued automaton $\mathfrak{R} = \langle Q, I, T, \delta\rangle$, *if* $|Q| = n$, then

$$\begin{aligned} \lceil A_{p_-} \equiv rec_{\mathfrak{R}}\rceil \leq {} & [A_{p_-}(\sigma^n\tau^n) \leftrightarrow rec_{\mathfrak{R}}(\sigma^n\tau^n)] \\ & \wedge \wedge_{k,l\in\omega \text{ s.t. } k\neq l}[A_{p_-}(\sigma^k\tau^l) \leftrightarrow rec_{\mathfrak{R}}(\sigma^k\tau^l)] \\ = {} & rec_{\mathfrak{R}}(\sigma^n\tau^n) \wedge \wedge_{k,l\in\omega \text{ s.t. } k\neq l}[p_- \leftrightarrow rec_{\mathfrak{R}}(\sigma^k\tau^l)] \end{aligned}$$

If $rec_{\mathfrak{R}}(\sigma^n\tau^n) \leq p_-$, then $\lceil A_{p_-} \equiv rec_{\mathfrak{R}}\rceil \leq p_-$. Now, we consider the case of $rec_{\mathfrak{R}}(\sigma^n\tau^n) \not\leq p_-$. For any $c \in T(Q, \Sigma)$, if $b(c) \in I$, $e(c) \in T$, and $lb(c) = \sigma^n\tau^n$, then c must be of the form $c = p_0\sigma p_1 \ldots p_{n-1}\sigma p_n\tau q_1 \ldots q_{n-1}\tau q_n$. Since $|Q| = n$, there are i, j such that $i < j \leq n$ and $p_i = p_j$. We put

$$\begin{aligned} c^+ = {} & p_0\sigma p_1 \ldots p_{j-1}\sigma p_j(\\ = {} & p_i)\sigma p_{i+1} \ldots p_{j-1}\sigma p_j\sigma p_{j+1} \ldots p_{n-1}\sigma p_n\tau q_1 \ldots q_{n-1}\tau q_n \end{aligned}$$

Then $b(c^+) \in I$, $e(c^+) \in T$, $lb(c^+) = \sigma^{n+(j-i)}\tau^n$, and $\lceil path_{\mathfrak{R}}(c^+)\rceil = \lceil path_{\mathfrak{R}}(c)\rceil$. Therefore, it holds that

$$\begin{aligned} rec_{\mathfrak{R}}(\sigma^{n+(j-i)}\tau^n) \geq {} & \vee\{\lceil path_{\mathfrak{R}}(c^+)\rceil\colon b(c) \in I, e(c) \in T \text{ and } lb(c) = \sigma^n\tau^n\} \\ = {} & \vee\{\lceil path_{\mathfrak{R}}(c)\rceil\colon b(c) \in I, e(c) \in T \text{ and } lb(c) = \sigma^n\tau^n\} \\ = {} & rec_{\mathfrak{R}}(\sigma^n\tau^n) \end{aligned}$$

and $rec_{\mathfrak{R}}(\sigma^{n+(j-i)}\tau^n) \not\leq p_-$. Furthermore, we have

$$\lceil A_{p_-} \equiv rec_{\Re}\rceil \leq p_- \leftrightarrow rec_{\Re}(\sigma^{n+(j-i)}\tau^n) \leq p_-$$

So, for all l-valued automata $\Re$ we have $\lceil A_{p_-} \equiv rec_{\Re}\rceil \leq p_-$, and it follows that $\lceil Rec_{\Sigma}(A_{p_-})\rceil = \vee\{\lceil A \equiv rec_{\Re}\rceil: \Re \in \mathbf{A}(\Sigma, l)\} \leq p_-$. This, together with $\lceil Rec_{\Sigma}(A_{p_-})\rceil \geq p_-$ obtained before, leads to $\lceil Rec_{\Sigma}(A_{p_-})\rceil = p_-$.

Similarly, we have $\lceil Rec_{\Sigma}(A_t)\rceil = t$ for $t = p_+$, $\overline{p_-}$, and $\overline{p_+}$.

Motivated by the above example, we propose the following:

Open Problem 3.5. Describe orthomodular lattices $l = \langle L, \leq, \wedge, \vee, \perp, 0, 1\rangle$ which satisfy $\{\lceil Rec_{\Sigma}(A)\rceil: A \in L^{\Sigma^*} = L$, i.e., the truth values of recognizability traverse all over L, or more explicitly, for every $\lambda \in L$, there is $A \in L^{\Sigma^*}$ such that $\lceil Rec_{\Sigma}(A)\rceil = \lambda$.

It seems that this is a difficult problem. Following are three more properties of Rec_{Σ}.

Proposition 3.6. For any $A \in L^{\Sigma^*}$, if A is finite, ie., *suppA* is finite, then $\overset{l}{\models} Rec_{\Sigma}(A)$.

Proof. Suppose that $suppA = \{\sigma_{i1} \ldots \sigma_{im_i}: i = 1, \ldots, k\}$. Then we construct an l-valued automaton $\Re_A = (Q_A, I_A, T_A, \delta_A)$ in the following way:

(i) $Q_A = \cup_{i=1}^{k}\{q_{i0}, q_{i1}, \ldots, q_{im_i}\}$.

(ii) $I_A = \{q_{10}, q_{20}, \ldots, q_{k0}\}$

(iii) $T_A = \{q_{1m_1}, q_{2m_2}, \ldots, q_{km_k}\}$.

(iv) We define $\delta_A(q_{ij}, \sigma_{i(j+1)}, q_{i(j+1)}) = A(\sigma_{i1} \ldots \sigma_{im_i})$ for any $1 \leq i \leq k$ and $0 \leq j < m_i$, and we define $\delta_A(p, \sigma, q) = 0$ for other $(p, \sigma, q) \in Q_A \times \Sigma \times Q_A$. Then it is easy to see that $rec_{\Re_A} = A$ and $\lceil Rec_{\Sigma}(A)\rceil \geq \lceil A \equiv rec_{\Re_A}\rceil = 1$. ■

For any $A \in L^{\Sigma^*}$, we define

$$A \downarrow \lambda = \{s \in \Sigma^*: A(s) \not\leq \lambda\}, \qquad A \uparrow \lambda = \{s \in \Sigma^*: A(s) \not\geq \lambda\}$$

Proposition 3.7. Let $A \in L^{\Sigma^*}$. Then:

(1) $\overset{l}{\models} \boldsymbol{\mu} \rightarrow Rec_{\Sigma}(A)$, where $\mu = \vee\{\lambda^{\perp}: A \downarrow \lambda$ is finite$\}$.

(2) $\overset{l}{\models} \boldsymbol{\theta} \rightarrow Rec_{\Sigma}(A)$, where $\theta = \vee\{\lambda: A \uparrow \lambda$ is finite$\}$.

Proof. We only prove (1); part (2) may be proven similarly.

For any $\lambda \in L$, if $A \downarrow \lambda$ is finite, then we define $A \Downarrow \lambda \in L^{\Sigma^*}$ as follows: for any $s \in \Sigma^*$,

$$(A \Downarrow \lambda)(s) = \begin{cases} A(s) & \text{if} \quad A(s) \not\leq \lambda, \\ 0 & \text{if} \quad A(s) \leq \lambda \end{cases}$$

Clearly, $A \Downarrow \lambda$ is finite. Then, from the proof of Proposition 2.6 we know that there is an l-valued automata $\Re[\lambda]$ such that $rec_{\Re[\lambda]} = A \Downarrow \lambda$, i.e., $rec_{\Re[\lambda]} = A(s)$ if $A(s) \not\leq \lambda$ and $rec_{\Re[\lambda]} = 0$ if $A(s) \leq \lambda$, and

$$\begin{aligned}\lceil Rec_\Sigma(A)\rceil \geq \lceil A \equiv rec_{\Re[\lambda]}\rceil &= \wedge\{A(s) \leftrightarrow rec_{\Re[\lambda]}: A(s) \not\leq \lambda\}\\ &\quad \wedge \wedge\{A(s) \leftrightarrow 0: A(s) \leq \lambda\}\\ &= \wedge\{A(s) \leftrightarrow 0: A(s) \leq \lambda\} \geq \lambda^\perp \quad \blacksquare\end{aligned}$$

Let $A \in L^{\Sigma^*}$. Then the inverse $A^{-1} \in L^{\Sigma^*}$ of A is defined as follows:

$$A^{-1}(\sigma_1 \ldots \sigma_m) = A(\sigma_m \ldots \sigma_1)$$

for any $m \in \omega$ and for any $\sigma_1, \ldots, \sigma_m \in \Sigma$.

Proposition 3.8. For any $A \in L^{\Sigma^*}$, $\overset{l}{\models} Rec_\Sigma(A) \leftrightarrow Rec_\Sigma(A^{-1})$.

Proof. Noting that $A = (A^{-1})^{-1}$, it suffices to show that $\lceil Rec_\Sigma(A)\rceil \leq \lceil Rec_\Sigma(A^{-1})\rceil$. For any l-valued automaton $\Re = (Q, I, T, \delta)$, we define the inverse of $\Re$ to be the l-valued automaton $\Re^{-1} = (Q, T, I, \delta^{-1})$, where $\delta^{-1}(p, \sigma, q) = \delta(q, \sigma, p)$ for any $p, q \in Q$ and $\sigma \in \Sigma$. Then it is easy to see that $rec_{\Re-1} = (rec_\Re)^{-1}$, and furthermore we have

$$\begin{aligned}\lceil Rec_\Sigma(A)\rceil &= \vee\{\lceil A \equiv rec_\Re\rceil: \Re \in \mathbf{A}(\Sigma, l)\}\\ &= \vee\{\lceil A^{-1} \equiv (rec_\Re)^{-1}\rceil: \Re \in \mathbf{A}(\Sigma, l)\}\\ &= \vee\{\lceil A^{-1} \equiv rec_{\Re^{-1}}\rceil: \Re \in \mathbf{A}(\Sigma, l)\}\\ &\leq \vee\{\lceil A^{-1} \equiv rec_{\mathfrak{p}}\rceil: \mathfrak{p} \in \mathbf{A}(\Sigma, l)\} = \lceil Rec_\Sigma(A^{-1})\rceil \quad \blacksquare\end{aligned}$$

4. CONCLUSION

In this paper, we outlined a framework of automata theory based on quantum logic, defined the orthomodular-valued (quantum) predicate of recognizabilty in automata theory, and established some of its basic properties. We tried to pursue more properties of quantum recognizability, but in doing so we found that the proof of some even very basic properties of automata (by the semantic analysis method) requires an essential application of the distributivity for the lattice of truth values of the underlying logic. This suggests the inverse problem: Which properties of automata require the distributivity of the lattice of truth values of the underlying logic? In other words, which properties of automata hold only in Boolean logic, but not in quantum logic? This is a very important problem for further study. If we can find some properties of this kind, then we know that although these properties may have been successfully used in the development of classical computer systems, they do not apply to quantum computers, and some new laws suitable for quantum computers have to be found.

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